Algorithmic Game Theory

Kevin Binz
Senja Filipi

Prisoner's Dilemma

		Person A	
		Cooperate	**Defect**
Person B	**Cooperate**	A: 1 year B: 1 year	A: 0 years B: 3 years
	Defect	A: 3 years B: 0 years	A: 2 years B: 2 years

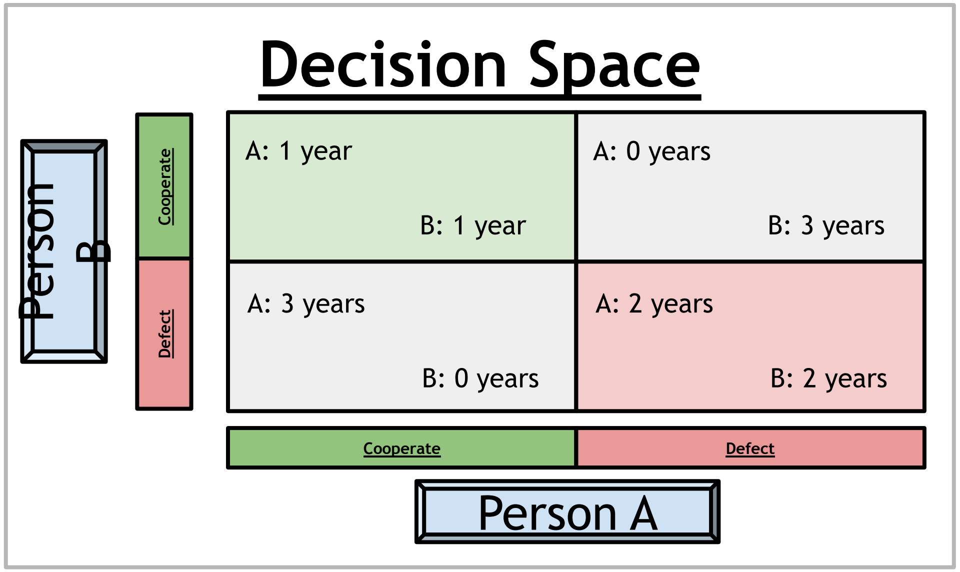
A's Dominant Strategy

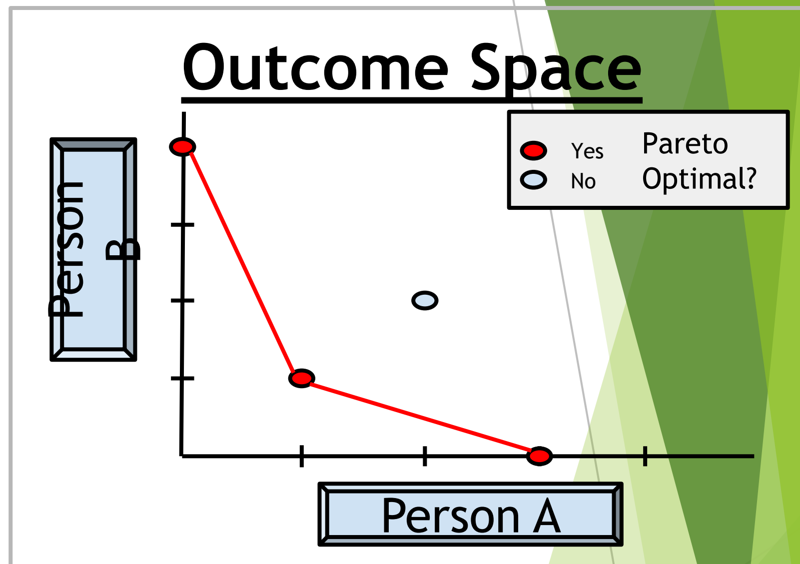
		Person A	
		Cooperate	**Defect**
Person B	**Cooperate**	A: 1 year B: 1 year	A: 0 years B: 3 years
	Defect	A: 3 years B: 0 years	A: 2 years B: 2 years

B’s Dominant Strategy

		Person A	
		Cooperate	**Defect**
Person B	**Cooperate**	A: 1 year B: 1 year	A: 0 years B: 3 years
	Defect	A: 3 years B: 0 years	A: 2 years B: 2 years

Pareto Optimality





The strategically dominant outcome (D, D) does not lie on the Pareto frontier.

Nash Equilibria: Spousal Game

Both spouses prefer each others company.

Failing that, they'd prefer to work late vs. staying home alone.

...No strategic dominance!

What happens when we look at *regret*?

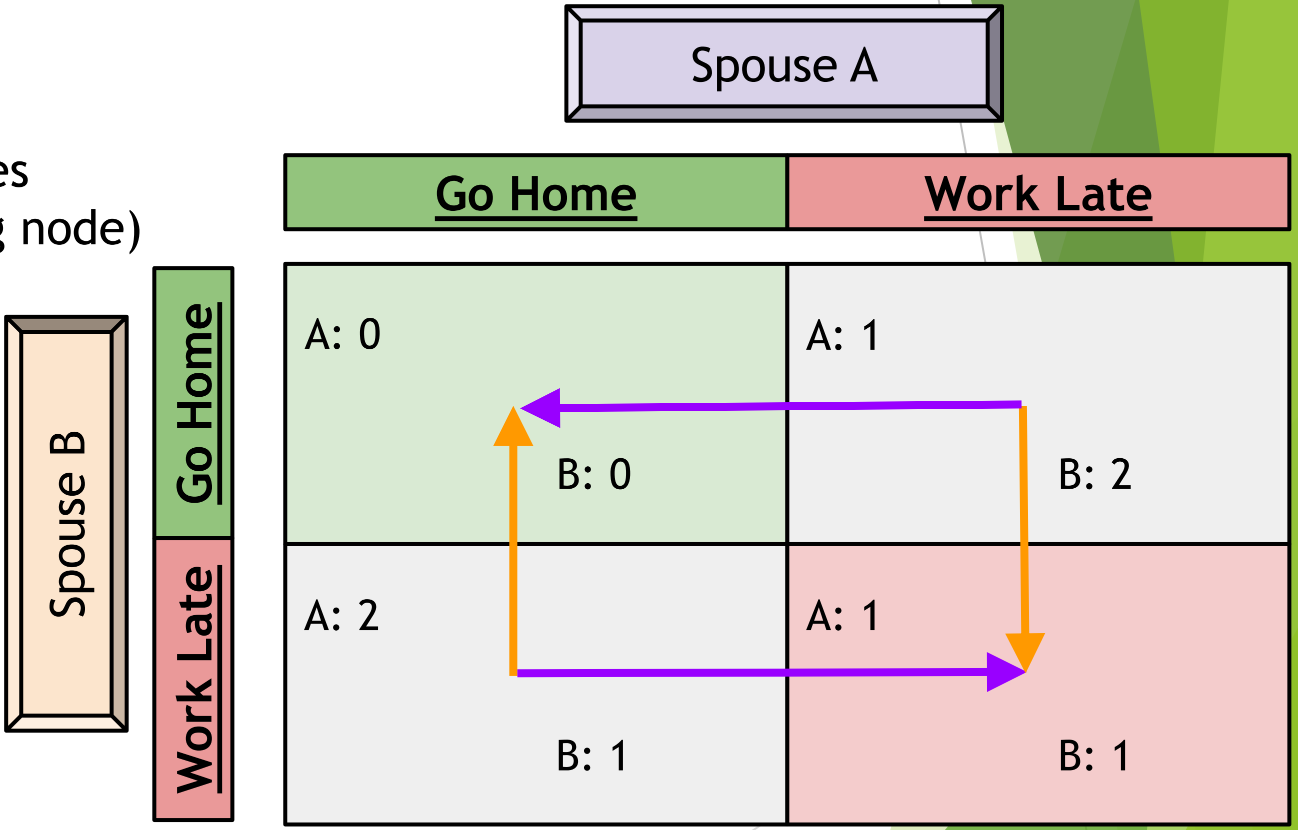
		Spouse A	
		Go Home	**Work Late**
Spouse B	**Go Home**	A: 0 B: 0	A: 1 B: 2
	Work Late	A: 2 B: 1	A: 1 B: 1

Nash Equilibria: Regret

In gray nodes, both spouses feel regret (arrows exiting node)

Nodes w/ zero regret are **Nash Equilibria**

This example has two equilibra.



The Importance Of Equilibria

Not every game features strategic dominance.

- Prisoner's Dilemma did, the Spousal Game did not.

Theorem: every finite game features at least one Nash equilibrium.

- Over time, rational players will tend to gravitate towards one of these equilibria, even if the utility topology is not perfectly known.

Sometimes we want to compare "selfish" outcomes with "designed" outcomes.

- Let **Price of Anarchy** refer to the ratio btw worst equilibrium & best outcome
- Let **Price Of Stability** refer to the ratio btw best equilibrium & best outcome.

Nash Equilibria: Rock/Paper/Scissors

Does this example have equilibria?

… Nope!

So it looks like we have already disproven our theorem…

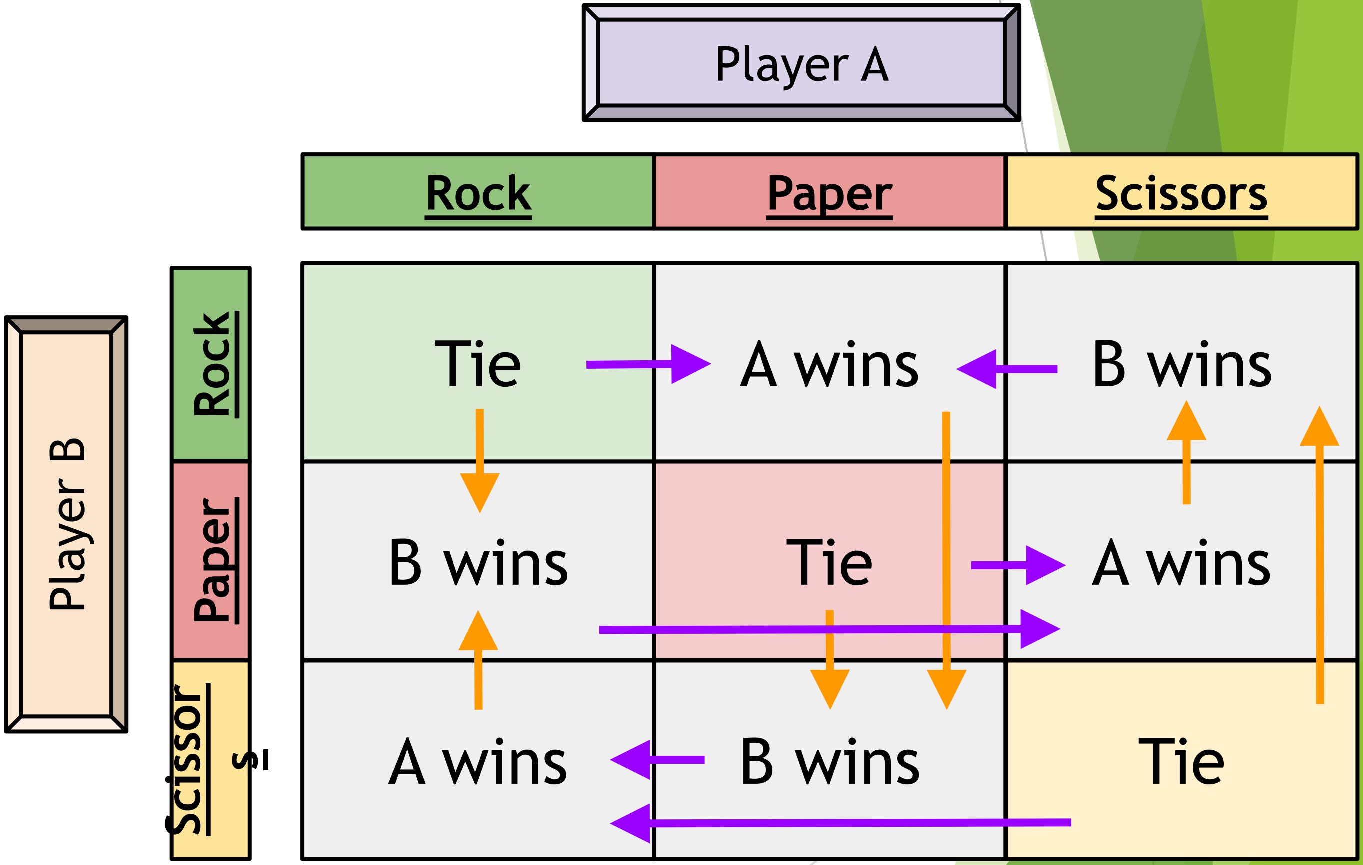
Player B \ Player A	**Rock**	**Paper**	**Scissors**
Rock	Tie	A wins	B wins
Paper	B wins	Tie	A wins
Scissors	A wins	B wins	Tie

Nash Equilibria: Mixed Strategies

We see no deterministic Nash equilibria, but having such a "pure" strategy in R/P/S is a bad idea…

What happens when we allow players to adopt "mixed" (non-deterministic) strategies?

The mixed strategy [⅓, ⅓, ⅓] produces a Nash equilibria!



Potential function

- Potential function – a function that captures the incentive of the players to change strategies.
  Sum of the utilities

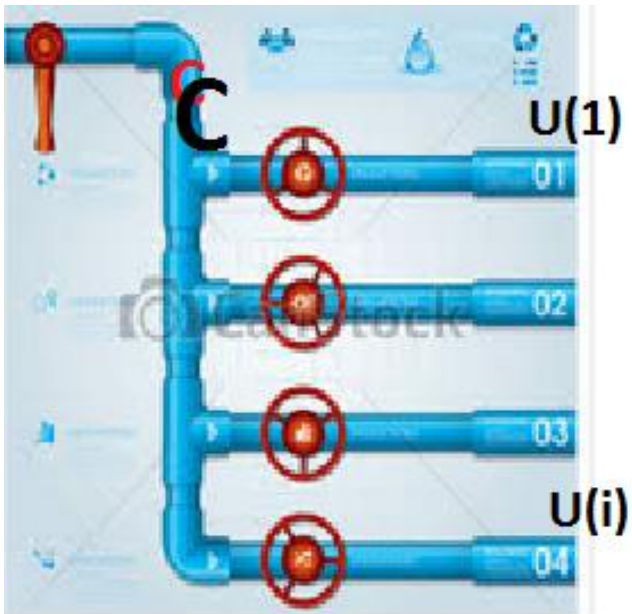
$$\Phi(S) - \Phi(S') = u_i(S') - u_i(S)$$

- Potential game – game characterized by a potential function.
- Nash equilibrium relation – local optima of the potential function

Example – Resource allocation

$$(U_1, \ldots, U_n, C)$$

U(i)



b_i

$$x_i = \frac{b_i}{\sum_{j=1}^{n} b_j} \cdot C$$

Efficiency
Optimal allocation
Inefficiency arises.

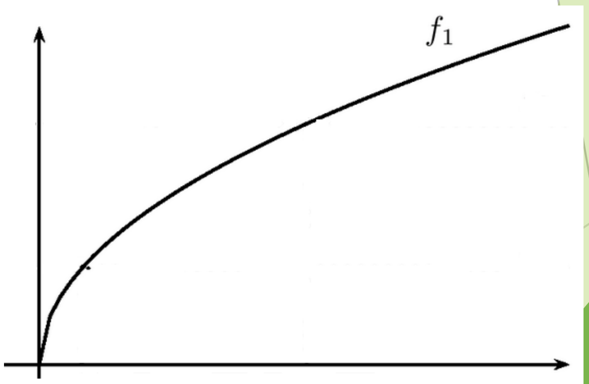
$$Q_i(b_1, \ldots, b_n) = U_i(x_i) - b_i = U_i\left(\frac{b_i}{\sum_{j=1}^{n} b_j} \cdot C\right) - b_i.$$

Potential function on resource allocation

$$\Phi_{RA}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \hat{U}_i(x_i),$$

$$\hat{U}_i(x_i) = \left(1 - \frac{x_i}{C}\right) \cdot U_i(x_i) + \frac{x_i}{C} \cdot \left(\frac{1}{x_i} \int_0^{x_i} U_i(y)dy\right).$$

- The function is
  - strictly concave
  - increasing
  - continuously differentiable.
- The above shows the equilibrium exists and it is unique.



Price of anarchy bound analysis

$$\frac{U(\hat{x}) + \hat{U}'(\hat{x})(x^* - \hat{x})}{U(x^*)}.$$

- Lower bound on the price of anarchy is ¾
- Utility in equilibrium: $\hat{U}'(\hat{x}) = U'(\hat{x}) \cdot [1 - (\hat{x}/C)].$
- Optimal allocation x* = C

$$\begin{aligned}
U(\hat{x}) + \hat{U}'(\hat{x})(x^* - \hat{x}) &= U(\hat{x}) + \left(1 - \frac{\hat{x}}{x^*}\right) U'(\hat{x})(x^* - \hat{x}) \\
&\geq U(\hat{x}) + \left(1 - \frac{\hat{x}}{x^*}\right)(U(x^*) - U(\hat{x})) \\
&= \left(\frac{\hat{x}}{x^*}\right) \cdot U(\hat{x}) + \left(1 - \frac{\hat{x}}{x^*}\right) \cdot U(x^*) \\
&\geq \left(\frac{\hat{x}}{x^*}\right)^2 \cdot U(x^*) + \left(1 - \frac{\hat{x}}{x^*}\right) \cdot U(x^*) \\
&\geq \frac{3}{4} \cdot U(x^*),
\end{aligned}$$

Potential games

- Characteristics
  - Rosenthal's Theorem - Pure equilibria always exists
  - Best response dynamics converges to Nash Equilibria
  - Price of stability can be bounded using the potential function technique

Types of potential games

When the strategy changes:

- Exact – delta of potential function equals delta in utility
- Weighted - delta of potential function when strategy changes is a portion of the delta in utility
- Ordinal – a positive change in the potential function guarantees an increase in utility, and vice-versa.
- Generalized - a positive change in the potential function guarantees an increase in utility
- Best response – players aim at maximizing each parties profit.

Other classical examples:

- Routing Games
- Location Games

References

- Algorithmic Game theory – Chapters 1, 2, 17 – 19.
- http://theory.stanford.edu/~tim/papers/ineff.pdf
- http://theory.stanford.edu/~tim/papers/icm.pdf
- http://theory.stanford.edu/~tim/papers/et.pdf
- http://www.mit.edu/people/jnt/Papers/J097-04-joh-ncgame.pdf
- http://www.library.fa.ru/files/Shapley4.pdf
- http://theory.stanford.edu/~tim/f13/l/l1.pdf - l20.pdf
- http://montoya.econ.ubc.ca/Econ522/Congestion_games_-_Rosenthal.pdf
- http://web.mit.edu/linguistics/events/iap07/Nash-Eqm.pdf